A GROSZEK - LAVER PAIR OF UNDISTINGUISHABLE E_0 CLASSES

MOHAMMAD GOLSHANI, VLADIMIR KANOVEI, AND VASSILY LYUBETSKY

Abstract. A generic extension $\mathbf{L}[x,y]$ of $\mathbf{L}$ by reals x,y is defined, in which the union of E_0-classes of x and y is a Π^1_2 set, but neither of these two E_0-classes is separately ordinal-definable.

1. Introduction

Let a *Groszek - Laver pair* be any unordered OD (ordinal-definable) pair $\{X,Y\}$ of sets $X,Y \subseteq \omega^\omega$ such that neither of X,Y is separately OD. As demonstrated in [3], if $\langle x,y\rangle$ is a Sacks$\times$Sacks generic pair of reals over $\mathbf{L}$, the constructible universe, then their degrees of constructibility $X=[x]_{\mathbf{L}}\cap\omega^\omega$ and $Y=[y]_{\mathbf{L}}\cap\omega^\omega$ form such a pair in $\mathbf{L}[x,y]$; the set $\{X,Y\}$ is definable as the set of all $\mathbf{L}$-degrees of reals, $\mathbf{L}$-minimal over $\mathbf{L}$.

As the sets X,Y in this example are obviously uncountable, one may ask whether there can consistently exist a Groszek – Laver pair of *countable* sets. The next theorem answers this question in the positive in a rather strong way: both sets are E_0-classes in the example! (Recall that the equivalence relation E_0 is defined on 2^ω as follows: $x\ \mathsf{E}_0\ y$ iff $x(n)=y(n)$ for all but finite n.)

Theorem 1.1. *It is true in a suitable generic extension $\mathbf{L}[x,y]$ of $\mathbf{L}$, by a pair of reals $x,y\in 2^\omega$ that the union of E_0-equivalence classes $[x]_{\mathsf{E}_0}\cup[y]_{\mathsf{E}_0}$ is Π^1_2, but neither of the sets $[x]_{\mathsf{E}_0},[y]_{\mathsf{E}_0}$ is separately OD.*

The forcing we employ is a conditional product $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ of an "E_0-large tree" [1] version $\mathbb{P}$ of a forcing notion, introduced in [12] to define a model with a Π^1_2 E_0-class containing no OD elements. The forcing in [12] was a clone of Jensen's minimal Π^1_2 real singleton forcing [7] (see also Section 28A of [6]), but defined on the base of the Silver forcing instead of the Sacks forcing. The crucial advantage of Silver's forcing here is that it leads to a

---

Date: March 07, 2015.

2000 *Mathematics Subject Classification.* Primary 03E15, 03E35; Secondary 03E45.

Key words and phrases. Forcing, equivalence classes, ordinal definability, Groszek - Laver pair.

The first author was supported in part by IPM Grant #91030417.

The second author was supported in part by RFBR Grant #13-01-00006.

The second and third authors were supported in part by RNF Grant #14-50-00150.

[1] An E_0-large tree is a perfect tree $T\subseteq 2^{<\omega}$ such that $\mathsf{E}_0\restriction[T]$ is not smooth, see [9, 10.9].

Jensen-type forcing naturally closed under the 0-1 flip at any digit, so that the corresponding extension contains a Π^1_2 E_0-class of generic reals instead of a Π^1_2 generic singleton as in [7].

In another relevant note [11] it is demonstrated that a countable OD set of reals (not an E_0-class), containing no OD elements, exists in a generic extension of $\mathbf{L}$ via the countable finite-support product of Jensen's [7] forcing itself. The existence of such a set was discussed as an open question at the *Mathoverflow* website [2] and at FOM [3], and the result in [11] was conjectured by Enayat (Footnote 3) on the base of his study of finite-support products of Jensen's forcing in [2].

The remainder of the paper is organized as follows.

We introduce E_0-large perfect trees in $2^{<\omega}$ in Section 2, study their splitting properties in Section 3, and consider E_0-large-tree forcing notions in Section 4, *i.e.*, collections of E_0-large trees closed under both restriction and action of a group of transformations naturally associated with E_0.

If $\mathbb{P}$ is an E_0-large-tree forcing notion then the *conditional product forcing* $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ is a part of the full forcing product $\mathbb{P}\times\mathbb{P}$ which contains all conditions $\langle T,T'\rangle$ of trees $T,T'\in\mathbb{P}$, E_0-connected in some way. This key notion, defined in Section 5, goes back to early research on the Gandy – Harrington forcing [5, 4].

The basic E_0-large-tree forcing $\mathbb{P}$ employed in the proof of Theorem 1.1 is defined, in $\mathbf{L}$, in the form $\mathbb{P}=\bigcup_{\xi<\omega_1}\mathbb{U}_\xi$ in Section 10. The model $\mathbf{L}[x,y]$ which proves the theorem is then a $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-generic extension of $\mathbf{L}$; it is studied in Section 11. The elements $\mathbb{U}_\xi$ of this inductive construction are countable E_0-large-tree forcing notions in $\mathbf{L}$.

The key issue is, given a subsequence $\{\mathbb{U}_\eta\}_{\eta<\xi}$ and accordingly the union $\mathbb{P}_{<\xi}=\bigcup_{\eta<\xi}\mathbb{U}_\eta$, to define the next level $\mathbb{U}_\xi$. We maintain this task in Section 7 with the help of a well-known splitting/fusion construction, modified so that it yields E_0-large perfect trees. Generic aspects of this construction lead to the CCC property of $\mathbb{P}$ and $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ and very simple reading of real names, but most of all to the crucial property that if $\langle x,y\rangle$ is a pair of reals $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-generic over $\mathbf{L}$ then any real $z\in\mathbf{L}[x,y]$ $\mathbb{P}$-generic over $\mathbf{L}$ belongs to $[x]_{\mathsf{E}_0}\cup[y]_{\mathsf{E}_0}$. This is Lemma 11.4 proved, on the base of preliminary results in Section 9.

The final Section 12 briefly discusses some related topics.

2. E_0-large trees

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers $0,1$, including the empty string Λ. If $t\in 2^{<\omega}$ and $i=0,1$ then $t^\frown i$ is the extension of t by i as the rightmost term. If $s,t\in 2^{<\omega}$ then $s\subseteq t$ means that t extends

[2] A question about ordinal definable real numbers. *Mathoverflow*, March 09, 2010. `http://mathoverflow.net/questions/17608`.

[3] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. `http://cs.nyu.edu/pipermail/fom/2010-July/014944.html`

s, $s \subset t$ means proper extension, and $s^\frown t$ is the concatenation. If $s \in 2^{<\omega}$ then $\mathtt{lh}(s)$ is the length of s, and we let $2^n = \{s \in 2^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length n).

Let any $s \in 2^{<\omega}$ **act** on 2^ω so that $(s \cdot x)(k) = x(k) + s(k) \pmod 2$ whenever $k < \mathtt{lh}(s)$ and simply $(s \cdot x)(k) = x(k)$ otherwise. If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$ then, as usual, let $s \cdot X = \{s \cdot x : x \in X\}$.

Similarly if $s, t \in 2^{<\omega}$ and $\mathtt{lh}(s) = m \le n = \mathtt{lh}(t)$, then define $s \cdot t \in 2^n$ so that $(s \cdot t)(k) = t(k) + s(k) \pmod 2$ whenever $k < m$ and $(s \cdot t)(k) = t(k)$ whenever $m \le k < n$. If $m > n$ then let simply $s \cdot t = (s \restriction n) \cdot t$. Note that $\mathtt{lh}(s \cdot t) = \mathtt{lh}(t)$ in both cases. Let $s \cdot T = \{s \cdot t : t \in T\}$ for $T \subseteq 2^{<\omega}$.

If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T \restriction_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let $\mathbf{PT}$ be the set of all *perfect* trees $\varnothing \ne T \subseteq 2^{<\omega}$ (those with no endpoints and no isolated branches). If $T \in \mathbf{PT}$ then there is a largest string $s \in T$ such that $T = T \restriction_s$; it is denoted by $s = \mathtt{stem}(T)$ (the *stem* of T); we have $s^\frown 1 \in T$ and $s^\frown 0 \in T$ in this case. If $T \in \mathbf{PT}$ then

$$[T] = \{a \in 2^\omega : \forall\, n\, (a \restriction n \in T)\} \subseteq 2^\omega$$

is the perfect set of all *paths through* T; clearly $[S] \subseteq [T]$ iff $S \subseteq T$.

Let $\mathbf{LT}$ (large trees) be the set of all *special* E_0*-large trees*: those $T \in \mathbf{PT}$ such that there is a double sequence of non-empty strings $q^i_n = q^i_n(T) \in 2^{<\omega}$, $n < \omega$ and $i = 0, 1$, such that

- $\mathtt{lh}(q^0_n) = \mathtt{lh}(q^1_n) \ge 1$ and $q^i_n(0) = i$ for all n;
- T consists of all substrings of strings of the form $r^\frown q_0^{i(0)\frown} q_1^{i(1)\frown} \dots {}^\frown q_n^{i(n)}$ in $2^{<\omega}$, where $r = \mathtt{stem}(T)$, $n < \omega$, and $i(0), i(1), \dots, i(n) \in \{0, 1\}$.

We let $\mathbf{spl}_0(T) = \mathtt{lh}(r)$ and then by induction $\mathbf{spl}_{n+1}(T) = \mathbf{spl}_n(T) + \mathtt{lh}(q^i_n)$, so that $\mathbf{spl}(T) = \{\mathbf{spl}_n(T) : n < \omega\} \subseteq \omega$ is the set of *splitting levels* of T. Then

$$[T] = \{a \in 2^\omega : a \restriction \mathtt{lh}(r) = r \wedge \forall\, n\, \big(a \restriction [\mathbf{spl}_n(T), \mathbf{spl}_{n+1}(T)) = q^0_n \text{ or } q^1_n\big)\}.$$

Lemma 2.1. *Assume that* $T \in \mathbf{LT}$ *and* $h \in \mathbf{spl}(T)$*. Then*

(i) *if* $u, v \in 2^h \cap T$ *then* $T \restriction_v = (u \cdot v) \cdot T \restriction_u$ *and* $(u \cdot v) \cdot T = T$;

(ii) *if* $\sigma \in 2^{<\omega}$ *then* $T = \sigma \cdot T$ *or* $T \cap (\sigma \cdot T)$ *is finite.*

Proof. (ii) Suppose that $T \cap (\sigma \cdot T)$ is infinite. Then there is an infinite branch $x \in [T]$ such that $y = \sigma \cdot x \in [T]$, too. We can assume that $\mathtt{lh}(\sigma)$ is equal to some $h = \mathbf{spl}_n(T)$. (If $\mathbf{spl}_{n-1}(T) < h < \mathbf{spl}_n(T)$ then extend σ by $\mathbf{spl}_n(T) - h$ zeros.) Then $\sigma = (x \restriction h) \cdot (y \restriction h)$. It remains to apply (i). $\square$

Example 2.2. If $s \in 2^{<\omega}$ then $T[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subset s\}$ is a tree in $\mathbf{LT}$, $\mathtt{stem}(T[s]) = s$, and $q^i_n(T[s]) = \langle i \rangle$ for all n, i. Note that $T[\Lambda] = 2^{<\omega}$ (the full binary tree), and $T[\Lambda] \restriction_s = (2^{<\omega}) \restriction_s = T[s]$ for all $s \in 2^{<\omega}$. $\square$

3. Splitting of large trees

The *simple splitting* of a tree $T \in \mathbf{LT}$ consists of smaller trees

$$T(\to 0) = T\restriction_{\mathtt{stem}(T)^\frown 0} \quad \text{and} \quad T(\to 1) = T\restriction_{\mathtt{stem}(T)^\frown 1}\,,$$

so that $[T(\to i)] = \{x \in [T] : x(h) = i\}$, where $h = \mathbf{spl}_0(T) = \mathtt{lh}(\mathtt{stem}(T))$. Clearly $T(\to i) \in \mathbf{LT}$ and $\mathbf{spl}(T(\to i)) = \mathbf{spl}(T) \smallsetminus \{\mathbf{spl}_0(T)\}$.

Lemma 3.1. *If $R, S, T \in \mathbf{LT}$, $S \subseteq R(\to 0)$, $T \subseteq R(\to 1)$, $\sigma \in 2^{<\omega}$, $T = \sigma \cdot S$, and $\mathtt{lh}(\sigma) \le \mathtt{lh}(\mathtt{stem}(S)) = \mathtt{lh}(\mathtt{stem}(T))$ then $U = S \cup T \in \mathbf{LT}$, $\mathtt{stem}(U) = \mathtt{stem}(R)$, and $S = U(\to 0)$, $T = U(\to 1)$.* □

The splitting can be iterated, so that if $s \in 2^n$ then we define

$$T(\to s) = T(\to s(0))(\to s(1))(\to s(2)) \dots (\to s(n-1))\,.$$

We separately define $T(\to \Lambda) = T$, where Λ is the empty string as usual.

Lemma 3.2. *In terms of Example 2.2, $T[s] = (2^{<\omega})(\to s) = (2^{<\omega})\restriction_s$, $\forall\, s$. Generally if $T \in \mathbf{LT}$ and $2^n \subseteq T$ then $T(\to s) = T\restriction_s$ for all $s \in 2^n$.* □

If $T, S \in \mathbf{LT}$ and $n \in \omega$ then let $S \subseteq_n T$ (S *n-refines* T) mean that $S \subseteq T$ and $\mathbf{spl}_k(T) = \mathbf{spl}_k(S)$ for all $k < n$. In particular, $S \subseteq_0 T$ iff simply $S \subseteq T$. By definition if $S \subseteq_{n+1} T$ then $S \subseteq_n T$ (and $S \subseteq T$), too.

Lemma 3.3. *Suppose that $T \in \mathbf{LT}$, $n < \omega$, and $h = \mathbf{spl}_n(T)$. Then*

- (i) *$T = \bigcup_{s \in 2^n} T(\to s)$ and $[T(\to s)] \cap [T(\to t)] = \varnothing$ for all $s \ne t$ in 2^n;*
- (ii) *if $S \in \mathbf{LT}$ then $S \subseteq_n T$* **iff** *$S(\to s) \subseteq T(\to s)$ for all strings $s \in 2^{\leqslant n}$* **iff** *$S \subseteq T$ and $S \cap 2^h = T \cap 2^h$;*
- (iii) *if $s \in 2^n$ then $\mathtt{lh}(\mathtt{stem}(T(\to s))) = h$ and there is a string $u[s] \in 2^h \cap T$ such that $T(\to s) = T\restriction_{u[s]}$;*
- (iv) *if $u \in 2^h \cap T$ then there is a string $s[u] \in 2^n$ s.t. $T\restriction_u = T(\to s[u])$;*
- (v) *if $s_0 \in 2^n$ and $S \in \mathbf{LT}$, $S \subseteq T(\to s_0)$, then there is a unique tree $T' \in \mathbf{LT}$ such that $T' \subseteq_n T$ and $T'(\to s_0) = S$.*

Proof. (iii) Define $u[s] = \mathtt{stem}(T)^\frown q_0^{s(0)}(T)^\frown q_1^{s(1)}(T)^\frown \dots {}^\frown q_{n-1}^{s(n-1)}(T)$.

(iv) Define $s = s[u] \in 2^n$ by $s(k) = u(\mathbf{spl}_k(T))$ for all $k < n$.

(v) Let $u_0 = u[s_0] \in 2^h$. Following Lemma 2.1, define T' so that $T' \cap 2^h = T \cap 2^h$, and if $u \in T \cap 2^h$ then $T'\restriction_u = (u \cdot u_0) \cdot S$; in particular $T'\restriction_{u_0} = S$. □

Lemma 3.4 (fusion)**.** *Suppose that $\dots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ is an infinite decreasing sequence of trees in $\mathbf{LT}$. Then*

- (i) *$T = \bigcap_n T_n \in \mathbf{LT}$;*
- (ii) *if $n < \omega$ and $s \in 2^{n+1}$ then $T(\to s) = T \cap T_n(\to s) = \bigcap_{m \ge n} T_m(\to s)$.*

Proof. Both parts are clear, just note that $\mathbf{spl}(T) = \{\mathbf{spl}_n(T_n) : n < \omega\}$. □

4. Large-tree forcing notions

Let a *large-tree forcing notion* (**LTF**) be any set $\mathbb{P} \subseteq \mathbf{LT}$ such that

(4.1) if $u \in T \in \mathbb{P}$ then $T \restriction_u \in \mathbb{P}$;

(4.2) if $T \in \mathbb{P}$ and $s \in 2^{<\omega}$ then $s \cdot T \in \mathbb{P}$.

We'll typically consider **LTF**s $\mathbb{P}$ containing the full tree $2^{<\omega}$. In this case, $\mathbb{P}$ contains all trees $T[s]$ of Example 2.2 by Lemma 3.2.

Any **LTF** $\mathbb{P}$ can be viewed as a forcing notion (if $T \subseteq T'$ then T is a stronger condition), and then it adds a real in 2^ω.

If $\mathbb{P} \subseteq \mathbf{LT}$, $T \in \mathbf{LT}$, $n < \omega$, and all split trees $T(\to s)$, $s \in 2^n$, belong to $\mathbb{P}$, then we say that T is an *n-collage over* $\mathbb{P}$. Let $\mathbf{LC}_n(\mathbb{P})$ be the set of all trees $T \in \mathbf{LT}$ which are n-collages over $\mathbb{P}$, and $\mathbf{LC}(\mathbb{P}) = \bigcup_n \mathbf{LC}_n(\mathbb{P})$. Note that $\mathbf{LC}_n(\mathbb{P}) \subseteq \mathbf{LC}_{n+1}(\mathbb{P})$ by (4.1).

Lemma 4.1. *Assume that $\mathbb{P} \subseteq \mathbf{LT}$ is a **LTF** and $n < \omega$. Then*

(i) *if $T \in \mathbf{LT}$ and $s_0 \in 2^n$ then $T(\to s_0) \in \mathbb{P}$ iff $T \in \mathbf{LC}_n(\mathbb{P})$;*

(ii) *if $P \in \mathbf{LC}_n(\mathbb{P})$, $s_0 \in 2^n$, $S \in \mathbb{P}$, and $S \subseteq P(\to s_0)$, then there is a tree $Q \in \mathbf{LC}_n(\mathbb{P})$ such that $Q \subseteq_n P$ and $Q(\to s_0) = S$;*

(iii) *if $P \in \mathbf{LC}_n(\mathbb{P})$ and a set $D \subseteq \mathbb{P}$ is open dense in $\mathbb{P}$, then there is a tree $Q \in \mathbf{LC}_n(\mathbb{P})$ such that $Q \subseteq_n P$ and $Q(\to s) \in D$ for all $s \in 2^n$;*

(iv) *if $P \in \mathbf{LC}_n(\mathbb{P})$, $S, T \in \mathbb{P}$, $s, t \in 2^n$, $S \subseteq P(\to s^\frown 0)$, $T \subseteq P(\to t^\frown 1)$, $\sigma \in 2^{<\omega}$, and $T = \sigma \cdot S$, then there is a tree $Q \in \mathbf{LC}_{n+1}(\mathbb{P})$, $Q \subseteq_{n+1} P$, such that $Q(\to s^\frown 0) \subseteq S$ and $Q(\to t^\frown 1) \subseteq T$.*

Recall that a set $D \subseteq \mathbb{P}$ is *open dense* in $\mathbb{P}$ iff, 1st, if $S \in \mathbb{P}$ then there is a tree $T \in D$, $T \subseteq S$, and 2nd, if $S \in \mathbb{P}$, $T \in D$, and $S \subseteq T$, then $S \in D$, too.

Proof. (i) If $T \in \mathbf{LC}_n(\mathbb{P})$ then by definition $T(\to s_0) \in \mathbb{P}$. To prove the converse, let $h = \mathbf{spl}_n(T)$, and let $h[s] \in 2^h \cap T$ satisfy $T(\to s) = T \restriction_{u[s]}$ for all $s \in 2^n$ by Lemma 3.3(iii). If $T(\to s_0) \in \mathbb{P}$ then $T(\to s) = T \restriction_{u[s]} = (u[s] \cdot u[s_0]) \cdot T \restriction_{u[s]}$ by Lemma 2.1, so $T(\to s) \in \mathbb{P}$ by (4.2). Thus $T \in \mathbf{LC}_n(\mathbb{P})$.

(ii) By Lemma 3.3(v) there is a tree $Q \in \mathbf{LT}$ such that $Q \subseteq_n P$ and $Q(\to s_0) = S$. We observe that Q belongs to $\mathbf{LC}_n(\mathbb{P})$ by (i).

(iii) Apply (ii) consecutively 2^n times (all $s \in 2^n$).

(iv) We first consider the case when $t = s$. If $\mathtt{lh}(\sigma) \le L = \mathtt{lh}(\mathtt{stem}(S)) = \mathtt{lh}(\mathtt{stem}(T))$ then by Lemma 3.1 $U = S \cup T \in \mathbf{LT}$, $\mathtt{stem}(U) = \mathtt{stem}(P(\to s))$, and $U(\to 0) = S$, $U(\to 1) = T$. Lemma 3.3(v) yields a tree $Q \in \mathbf{LT}$ such that $Q \subseteq_n P$ and $Q(\to s) = U$, hence $\mathtt{stem}(Q(\to s)) = \mathtt{stem}(P(\to s))$ by the above. This implies $\mathbf{spl}_n(Q) = \mathbf{spl}_n(P)$ by Lemma 3.3(iii), and hence $Q \subseteq_{n+1} P$. And finally $Q \in \mathbf{LC}_{n+1}(\mathbb{P})$ by (i) since $Q(\to s^\frown 0) = S \in \mathbb{P}$.

Now suppose that $\mathtt{lh}(\sigma) > L$. Take any string $u \in S$ with $\mathtt{lh}(u) \geq \mathtt{lh}(s)$. The set $S' = S\restriction_u \subseteq S$ belongs to $\mathbb{P}$ and obviously $\mathtt{lh}(\mathtt{stem}(S')) \geq \mathtt{lh}(\sigma)$. It remains to follow the case already considered for the trees S' and $T' = \sigma \cdot S'$.

Finally consider the general case $s \neq t$. Let $h = \mathbf{spl}_n(P)$, $H = \mathbf{spl}_{n+1}(P)$. Let $u = u[s]$ and $v = u[t]$ be the strings in $P \cap 2^h$ defined by Lemma 3.3(iii) for P, so that $P\restriction_u = P(\to s)$ and $P\restriction_v = P(\to t)$, and let $U, V \in 2^H \cap P$ be defined accordingly so that $P\restriction_U = P(\to s^\frown 1)$ and $P\restriction_V = P(\to t^\frown 1)$. Let $\rho = u \cdot v$. Then $P(\to s) = \rho \cdot P(\to t)$ by Lemma 2.1. However we have $U = u^\frown \tau$ and $V = v^\frown \tau$ for one and the same string τ, see the proof of Lemma 3.3(iii). Therefore $U \cdot V = u \cdot v = \rho$ and $P(\to s^\frown 1) = \rho \cdot P(\to t^\frown 1)$ still by Lemma 2.1.

It follows that the tree $T_1 = \rho \cdot T$ satisfies $T_1 \subseteq P(\to s^\frown 1)$. Applying the result for $s = t$, we get a tree $Q \in \mathbf{LC}_{n+1}(\mathbb{P})$, $Q \subseteq_{n+1} P$, such that $Q(\to s^\frown 0) \subseteq S$ and $Q(\to s^\frown 1) \subseteq T_1$. Then by definition $\mathbf{spl}_k(P) = \mathbf{spl}_k(Q)$ for all $k \leq n$, and $Q(\to s) \subseteq P(\to s)$ for all $s \in 2^{n+1}$ by Lemma 3.3(ii). Therefore the same strings u, v satisfy $Q\restriction_u = Q(\to s)$ and $Q\restriction_v = Q(\to t)$. The same argument as above implies $Q(\to t^\frown 1) = \rho \cdot Q(\to s^\frown 1)$. We conclude that $Q(\to t^\frown 1) \subseteq \rho \cdot T_1 = T$, as required. □

5. Conditional product forcing

Along with any **LTF** $\mathbb{P}$, we'll consider the **conditional product** $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, which by definition consists of all pairs $\langle T, T' \rangle$ of trees $T, T' \in \mathbb{P}$ such that there is a string $s \in 2^{<\omega}$ satisfying $s \cdot T = T'$. We order $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ componentwise so that $\langle S, S' \rangle \leq \langle T, T' \rangle$ ($\langle S, S' \rangle$ is stronger) iff $S \subseteq T$ and $S' \subseteq T'$. [4]

Remark 5.1. $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces a pair of $\mathbb{P}$-generic reals. Indeed if $\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ with $s \cdot T = T'$ and $S \in \mathbb{P}$, $S \subseteq T$, then there is a tree $S' = s \cdot S \in \mathbb{P}$ (we make use of (4.2)) such that $\langle S, S' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ and $\langle S, S' \rangle \leq \langle T, T' \rangle$. □

But $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic pairs are not necessarily generic in the sense of the true forcing product $\mathbb{P} \times \mathbb{P}$. Indeed, if say $\mathbb{P}$ = Sacks (all perfect trees) then any $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$-generic pair $\langle x, y \rangle$ has the property that x, y belong to same E_0-invariant Borel sets coded in the ground universe, while for any uncountable and co-uncountable Borel set U coded in the ground universe there is a $\mathbb{P} \times \mathbb{P}$-generic pair $\langle x, y \rangle$ with $x \in U$ and $y \notin U$.

Lemma 5.2. *Assume that* $\mathbb{P}$ *is a* **LTF**, $n \geq 1$, $P \in \mathbf{LC}_n(\mathbb{P})$, *and a set* $D \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ *is open dense in* $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. *Then there is a tree* $Q \in \mathbf{LC}_n(\mathbb{P})$ *such that* $Q \subseteq_n P$ *and* $\langle Q(\to s), Q(\to t) \rangle \in D$ *whenever* $s, t \in 2^n$ *and* $s(n-1) \neq t(n-1)$.

[4] Conditional product forcing notions of this kind were considered in [5, 4, 8] and some other papers with respect to the Gandy – Harrington and similar forcings, and recently in [13] with respect to many forcing notions.

Proof (compare to Lemma 4.1(iii))**.** Let $s,t \in 2^n$ be any pair with $s(n-1) \ne t(n-1)$. By the density there is a condition $\langle S,T\rangle \in D$ such that $S \subseteq P(\to s)$ and $T \subseteq P(\to t)$. Note that $T = \sigma \cdot S$ for some $s \in 2^{<\omega}$ since $\langle S,T\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. Applying Lemma 4.1(iv) ($n+1$ there corresponds to n here) we obtain a tree $P' \in \mathbf{LC}_n(\mathbb{P})$ such that $P' \subseteq_n P$ and $P'(\to s) \subseteq S$, $P'(\to t) \subseteq T$. Then $\langle P'(\to s), P'(\to t)\rangle \in D$, as D is open. Consider all pairs $s,t \in 2^n$ with $s(n-1) \ne t(n-1)$ one by one. $\square$

Lemma 5.3. *Assume that* $\mathbb{P}$ *is a* **LTF**, $\langle T,T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $n<\omega$, $s,t \in 2^n$. *Then* $\langle T(\to s), T'(\to t)\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$.

Proof. Let $\sigma \in 2^{<\omega}$ satisfy $\sigma \cdot T = T'$. Note that $\mathbf{spl}(T) = \mathbf{spl}(T')$, hence we define $h = \mathbf{spl}_n(T) = \mathbf{spl}_n(T')$. By Lemma 3.3(iii), there are strings $u \in 2^h \cap T$ and $v \in 2^h \cap T'$ such that $T(\to s) = T\restriction_u$ and $T'(\to t) = T'\restriction_v$. Then obviously $\sigma \cdot T\restriction_u = T'\restriction_{v'}$, where $v' = \sigma \cdot u$. On the other hand $T'\restriction_v = (v \cdot v') \cdot T'\restriction_{v'}$ by Lemma 2.1. It follows that $T'\restriction_v = (v \cdot v' \cdot \sigma) \cdot T\restriction_u$, as required. $\square$

Corollary 5.4. *Assume that* $\mathbb{P}$ *is a* **LTF**. *Then* $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ *forces* $\dot{\boldsymbol{x}}_{\mathtt{left}} \not\mathsf{E}_0 \dot{\boldsymbol{x}}_{\mathtt{right}}$, *where* $\langle \dot{\boldsymbol{x}}_{\mathtt{left}}, \dot{\boldsymbol{x}}_{\mathtt{right}}\rangle$ *is a name of the* $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$*-generic pair.*

Proof. Otherwise a condition $\langle T,T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces $\dot{\boldsymbol{x}}_{\mathtt{right}} = \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{left}}$, where $\sigma \in 2^{<\omega}$. Find n and $s,t \in 2^n$ such that $T'(\to t) \cap (\sigma \cdot T(\to s)) = \varnothing$ and apply the lemma. $\square$

6. Multitrees

Let a *multitree* be any sequence $\varphi = \{\langle \tau_k^\varphi, h_k^\varphi\rangle\}_{k<\omega}$ such that

(6.1) if $k<\omega$ then $h_k^\varphi \in \omega \cup \{-1\}$, and the set $|\varphi| = \{k : h_k^\varphi \ne -1\}$ (the *support* of φ) is finite;

(6.2) if $k \in |\varphi|$ then $\tau_k^\varphi = \langle T_k^\varphi(0), T_k^\varphi(1), \dots, T_k^\varphi(h_k^\varphi)\rangle$, where each $T_k^\varphi(n)$ is a tree in **LT** and $T_k^\varphi(n) \subseteq_n T_k^\varphi(n-1)$ whenever $1 \le n \le h_k^\varphi$, while if $k \notin |\varphi|$ then simply $\tau_k^\varphi = \Lambda$ (the empty sequence).

In this context, if $n \le h_k^\varphi$ and $s \in 2^n$ then let $T_k^\varphi(s) = T_k^\varphi(n)(\to s)$.

Let φ, ψ be multitrees. Say that φ *extends* ψ, symbolically $\psi \preccurlyeq \varphi$, if $|\psi| \subseteq |\varphi|$, and, for every $k \in |\psi|$, we have $h_k^\varphi \ge h_k^\psi$ and τ_k^φ extends τ_k^ψ, so that $T_k^\varphi(n) = T_k^\psi(n)$ for all $n \le h_k^\psi$;

If $\mathbb{P}$ is a **LTF** then let $\mathbf{MT}(\mathbb{P})$ (*multitrees over* $\mathbb{P}$) be the set of all multitrees φ such that $T_k^\varphi(n) \in \mathbf{LC}_n(\mathbb{P})$ whenever $k \in |\varphi|$ and $n \le h_k^\varphi$.

7. Jensen's extension of a large-tree forcing notion

Let $\mathbf{ZFC}'$ be the subtheory of **ZFC** including all axioms except for the power set axiom, plus the axiom saying that $\mathscr{P}(\omega)$ exists. (Then ω_1, 2^ω, and sets like **PT** exist as well.)

Definition 7.1. Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$. Suppose that $\mathbb{P} \in \mathfrak{M}$, $\mathbb{P} \subseteq \mathbf{LT}$ is a $\mathbf{LTF}$. Then $\mathbf{MT}(\mathbb{P}) \in \mathfrak{M}$. A set $D \subseteq \mathbf{MT}(\mathbb{P})$ is *dense in* $\mathbf{MT}(\mathbb{P})$ iff for any $\psi \in \mathbf{MT}(\mathbb{P})$ there is a multitree $\varphi \in D$ such that $\psi \preccurlyeq \varphi$.

Consider any $\preccurlyeq$-increasing sequence $\Phi = \{\varphi(j)\}_{j<\omega}$ of multitrees

$$\varphi(j) = \{\langle \tau_k^{\varphi(j)}, h_k^{\varphi(j)} \rangle\}_{k<\omega} \in \mathbf{MT}(\mathbb{P}),$$

generic over $\mathfrak{M}$ in the sense that it intersects every set D, $D \subseteq \mathbf{MT}(\mathbb{P})$, dense in $\mathbf{MT}(\mathbb{P})$, which belongs to $\mathfrak{M}$. Then in particular Φ intersects every set

$$D_{kp} = \{\varphi \in \mathbf{MT}(\mathbb{P}) : k \in |\varphi| \wedge h_k^{\varphi} \geq p\}, \quad k, p < \omega.$$

Therefore if $k < \omega$ then by definition there is an infinite sequence

$$\ldots \subseteq_5 \boldsymbol{T}_k^{\Phi}(4) \subseteq_4 \boldsymbol{T}_k^{\Phi}(3) \subseteq_3 \boldsymbol{T}_k^{\Phi}(2) \subseteq_2 \boldsymbol{T}_k^{\Phi}(1) \subseteq_1 \boldsymbol{T}_k^{\Phi}(0)$$

of trees $\boldsymbol{T}_k^{\Phi}(n) \in \mathbf{LC}_n(\mathbb{P})$, such that, for any j, if $k \in |\varphi(j)|$ and $n \leq h_k^{\varphi(j)}$ then $T_k^{\varphi(j)}(n) = \boldsymbol{T}_k^{\Phi}(n)$. If $n < \omega$ and $s \in 2^n$ then we let $\boldsymbol{T}_k^{\Phi}(s) = \boldsymbol{T}_k^{\Phi}(n)(\to s)$; then $\boldsymbol{T}_k^{\Phi}(s) \in \mathbb{P}$ since $\boldsymbol{T}_k^{\Phi}(n) \in \mathbf{LC}_n(\mathbb{P})$. Then it follows from Lemma 3.4 that

$$\boldsymbol{U}_k^{\Phi} = \bigcap_n \boldsymbol{T}_k^{\Phi}(n) = \bigcap_n \bigcup_{s \in 2^n} \boldsymbol{T}_k^{\Phi}(s) \tag{1}$$

is a tree in $\mathbf{LT}$ (not necessarily in $\mathbb{P}$), as well as the trees $\boldsymbol{U}_k^{\Phi}(\to s)$, and still by Lemma 3.4,

$$\boldsymbol{U}_k^{\Phi}(\to s) = \boldsymbol{U}_k^{\Phi} \cap \boldsymbol{T}_k^{\Phi}(s) = \bigcap_{n \geq \mathtt{lh}(s)} \boldsymbol{T}_k^{\Phi}(n)(\to s) = \bigcap_{n \geq \mathtt{lh}(s)} \bigcup_{t \in 2^n,\, s \subseteq t} \boldsymbol{T}_k^{\Phi}(t), \tag{2}$$

and obviously $\boldsymbol{U}_k^{\Phi} = \boldsymbol{U}_k^{\Phi}(\to \Lambda)$.

Define a set of trees $\mathbb{U} = \{\sigma \cdot \boldsymbol{U}_k^{\Phi}(\to s) : k < \omega \wedge s \in 2^{<\omega} \wedge \sigma \in 2^{<\omega}\} \subseteq \mathbf{LT}$. □

The next few simple lemmas show useful effects of the genericity of Φ; their common motto is that the extension from $\mathbb{P}$ to $\mathbb{P} \cup \mathbb{U}$ is rather innocuous.

Lemma 7.2. *Both* $\mathbb{U}$ *and the union* $\mathbb{P} \cup \mathbb{U}$ *are* $\mathbf{LTF}$*s;* $\mathbb{P} \cap \mathbb{U} = \varnothing$.

Proof. To prove the last claim, let $T \in \mathbb{P}$ and $U = \boldsymbol{U}_k^{\Phi}(\to s) \in \mathbb{U}$. (If $U = \sigma \cdot \boldsymbol{U}_k^{\Phi}(\to s)$, $\sigma \in 2^{<\omega}$, then replace T by $\sigma \cdot T$.) The set $D(T, k)$ of all multitrees $\varphi \in \mathbf{MT}(\mathbb{P})$, such that $k \in |\varphi|$ and $T \smallsetminus T_k^{\varphi}(n)(\to s) \neq \varnothing$, where $n = h_k^{\varphi}$, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{MT}(\mathbb{P})$. Now any multitree $\varphi(j) \in D(T, k)$ witnesses that $T \smallsetminus \boldsymbol{U}_k^{\Phi}(\to s) \neq \varnothing$. □

Lemma 7.3. *The set* $\mathbb{U}$ *is dense in* $\mathbb{U} \cup \mathbb{P}$. *The set* $\mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$ *is dense in* $(\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$.

Proof. Suppose that $T \in \mathbb{P}$. The set $D(T)$ of all multitrees $\varphi \in \mathbf{MT}(\mathbb{P})$, such that $T_k^{\varphi}(0) = T$ for some k, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{MT}(\mathbb{P})$. It follows that $\varphi(j) \in D(T)$ for some j, by the choice of Φ. Then $\boldsymbol{T}_k^{\Phi}(\Lambda) = T$ for some k. However by construction $\boldsymbol{U}_k^{\Phi}(\to \Lambda) = \boldsymbol{U}_k^{\Phi} \subseteq \boldsymbol{T}_k^{\Phi}(\Lambda)$.

Now suppose that $\langle T,T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, so that $T' = \sigma \cdot T$, $\sigma \in 2^{<\omega}$. By Lemma 7.2 ($\mathbb{P} \cap \mathbb{U} = \varnothing$) it is impossible that one of the trees T, T' belongs to $\mathbb{P}$ and the other one to $\mathbb{U}$. Therefore we can assume that $T, T' \in \mathbb{P}$. By the first claim of the lemma, there is a tree $U \in \mathbb{U}$, $U \subseteq T$. Then $U' = \sigma \cdot U \in \mathbb{U}$ and still $U' = \sigma \cdot U$, hence $\langle U, U'\rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$, and it extends $\langle T, T'\rangle$. □

Lemma 7.4. *If $k, l < \omega$, $k \neq l$, and $\sigma \in 2^{<\omega}$ then $\boldsymbol{U}^{\Phi}_k \cap (\sigma \cdot \boldsymbol{U}^{\Phi}_l) = \varnothing$.*

Proof. The set $D'(k,l)$ of all multitrees $\varphi \in \mathbf{MT}(\mathbb{P})$, such that $k, l \in |\varphi|$ and $T^{\varphi}_k(n) \cap (\sigma \cdot T^{\varphi}_l(m)) = \varnothing$ for some $n \le h^{\varphi}_k$, $m \le h^{\varphi}_l$, belongs to $\mathfrak{M}$ and is dense in $\mathbf{MT}(\mathbb{P})$. So $\varphi(j) \in D'(k,l)$ for some $j < \omega$. But then for some n, m we have $\boldsymbol{U}^{\Phi}_k \cap (\sigma \cdot \boldsymbol{U}^{\Phi}_l) \subseteq T^{\varphi(j)}_k(n) \cap (\sigma \cdot T^{\varphi(j)}_l(m)) = \varnothing$. □

Corollary 7.5. *If $\langle U, U'\rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$ then there exist: $k < \omega$, strings $s, s' \in 2^{<\omega}$ with $\mathtt{lh}(s) = \mathtt{lh}(s')$, and strings $\sigma, \sigma' \in 2^{<\omega}$, such that $U = \sigma \cdot \boldsymbol{U}^{\Phi}_k(\to s)$ and $U' = \sigma' \cdot \boldsymbol{U}^{\Phi}_k(\to s')$.*

Proof. By definition, we have $U = \sigma \cdot \boldsymbol{U}^{\Phi}_k(\to s)$ and $U' = \sigma' \cdot \boldsymbol{U}^{\Phi}_{k'}(\to s')$, for suitable $k, k' < \omega$ and $s, s', \sigma, \sigma' \in 2^{<\omega}$. As $\langle U, U'\rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$, it follows from Lemma 7.4 that $k' = k$, hence $U' = \sigma \cdot \boldsymbol{U}^{\Phi}_k(\to s')$. Therefore $\sigma \cdot \boldsymbol{U}^{\Phi}_k(\to s) = \tau \cdot \sigma' \cdot \boldsymbol{U}^{\Phi}_k(\to s')$ for some $\tau \in 2^{<\omega}$. In other words, $\boldsymbol{U}^{\Phi}_k(\to s) = \tau' \cdot \boldsymbol{U}^{\Phi}_k(\to s')$, where $\tau' = \sigma \cdot \sigma' \cdot \tau \in 2^{<\omega}$. It easily follows that $\mathtt{lh}(s) = \mathtt{lh}(s')$. □

The two following lemmas show that, due to the generic character of extension, those pre-dense sets which belong to $\mathfrak{M}$, remain pre-dense in the extended forcing.

Let $X \subseteq^{\mathtt{fin}} \bigcup D$ mean that there is a finite set $D' \subseteq D$ with $X \subseteq \bigcup D'$.

Lemma 7.6. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ is pre-dense in $\mathbb{P}$, and $U \in \mathbb{U}$, then $U \subseteq^{\mathtt{fin}} \bigcup D$. Moreover D is pre-dense in $\mathbb{U} \cup \mathbb{P}$.*

Proof. We can assume that D is in fact open dense in $\mathbb{P}$. (Otherwise replace it with the set $D' = \{T \in \mathbb{P} : \exists\, S \in D\, (T \subseteq S)\}$ which also belongs to $\mathfrak{M}$.)

We can also assume that $U = \boldsymbol{U}^{\Phi}_k(\to s) \in \mathbb{U}$, where $k < \omega$ and $s \in 2^{<\omega}$. (The general case, when $U = \sigma \cdot \boldsymbol{U}^{\Phi}_k(\to s)$ for some $\sigma \in 2^{<\omega}$, is reducible to the case $U = \boldsymbol{U}^{\Phi}_k(\to s)$ by substituting the set $\sigma \cdot D$ for D.)

The set $\Delta \in \mathfrak{M}$ of all multitrees $\varphi \in \mathbf{MT}(\mathbb{P})$ such that $k \in |\varphi|$, $\mathtt{lh}(s) < h = h^{\varphi}_k$, and $T^{\varphi}_k(h)(\to t) \in D$ for all $t \in 2^h$, is dense in $\mathbf{MT}(\mathbb{P})$ by Lemma 4.1(iii) and the open density of D. Therefore there is an index j such that $\varphi(j) \in \Delta$. Let $h(j) = h^{\varphi(j)}_k$. Then the tree $S_t = T^{\varphi(j)}_k(h(j))(\to t) = \boldsymbol{T}^{\Phi}_k(h(j))(\to t) = \boldsymbol{T}^{\Phi}_k(t)$ belongs to D for all $t \in 2^{h(j)}$. We conclude that

$$U = \boldsymbol{U}^{\Phi}_k(\to s) \subseteq \boldsymbol{U}^{\Phi}_k \subseteq \bigcup\nolimits_{t \in 2^{h(j)}} \boldsymbol{T}^{\Phi}_k(t) \subseteq \bigcup\nolimits_{t \in 2^{h(j)}} S_t = \bigcup D',$$

where $D' = \{S_t : t \in 2^{h(j)}\} \subseteq D$ is finite.

To prove the pre-density claim, pick a string $t \in 2^{h(j)}$ with $s \subset t$. Then $V = \boldsymbol{U}^{\Phi}_k(\to t) \in \mathbb{U}$ and $V \subseteq U$. However $V \subseteq \boldsymbol{T}^{\Phi}_k(t) = S_t \in D$. Thus V witnesses that U is compatible with $S_t \in D$ in $\mathbb{U} \cup \mathbb{P}$, as required. □

Lemma 7.7. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is pre-dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ then D is pre-dense in $(\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$.*

Proof. Let $\langle U, U' \rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$; the goal is to prove that $\langle U, U' \rangle$ is compatible in $(\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$ with a condition $\langle T, T' \rangle \in D$. By Corollary 7.5, there exist: $k < \omega$ and strings $s, s', \sigma, \sigma' \in 2^{<\omega}$ such that $\mathtt{lh}(s) = \mathtt{lh}(s')$ and $U = \sigma \cdot \boldsymbol{U}_k^{\Phi}(\to s)$, $U' = \sigma' \cdot \boldsymbol{U}_k^{\Phi}(\to s')$. As in the proof of the previous lemma, we can assume that $\sigma = \sigma' = \Lambda$, so that $U = \boldsymbol{U}_k^{\Phi}(\to s)$, $U' = \boldsymbol{U}_k^{\Phi}(\to s')$. (The general case is reducible to this case by substituting the set $\{\langle \sigma \cdot T, \sigma' \cdot T' \rangle : \langle T, T' \rangle \in D\}$ for D.)

Assume that D is in fact open dense.

Consider the set $\Delta \in \mathfrak{M}$ of all multitrees $\varphi \in \mathbf{MT}(\mathbb{P})$ such that $k \in |\varphi|$, $\mathtt{lh}(s) = \mathtt{lh}(s') = n < h = h_k^{\varphi}$, and $\langle T_k^{\varphi}(h)(\to u), T_k^{\varphi}(h)(\to u') \rangle \in D$ whenever $u, u' \in 2^h$ and $u(h-1) \neq u'(h-1)$. The set Δ is dense in $\mathbf{MT}(\mathbb{P})$ by Lemma 5.2. Therefore $\varphi(j) \in \Delta$ for some j, so that if $u, u' \in 2^{h(j)}$, where $h(j) = h_k^{\varphi(j)} > n$, and $u(h(j) - 1) \neq u'(h(j) - 1)$, then

$$\langle T_k^{\varphi(j)}(h(j))(\to u), T_k^{\varphi(j)}(h(j))(\to u') \rangle = \langle \boldsymbol{T}_k^{\Phi}(u), \boldsymbol{T}_k^{\Phi}(u') \rangle \in D\,.$$

Now, as $h(j) > n$, let us pick $u, u' \in 2^{h(j)}$ such that $u(h(j)-1) \neq u'(h(j)-1)$ and $s \subset u$, $s' \subset u'$. Then $\langle \boldsymbol{T}_k^{\Phi}(u), \boldsymbol{T}_k^{\Phi}(u') \rangle \in D$. On the other hand, the pair $\langle \boldsymbol{U}_k^{\Phi}(\to u), \boldsymbol{U}_k^{\Phi}(\to u') \rangle$ belongs to $\mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$ by Lemma 5.3,

$$\langle \boldsymbol{U}_k^{\Phi}(\to u), \boldsymbol{U}_k^{\Phi}(\to u') \rangle \leq \langle \boldsymbol{U}_k^{\Phi}(\to s), \boldsymbol{U}_k^{\Phi}(\to s') \rangle\,,$$

and finally we have $\langle \boldsymbol{U}_k^{\Phi}(\to u), \boldsymbol{U}_k^{\Phi}(\to u') \rangle \leq \langle \boldsymbol{T}_k^{\Phi}(u), \boldsymbol{T}_k^{\Phi}(u') \rangle$. We conclude that the given condition $\langle \boldsymbol{U}_k^{\Phi}(\to s), \boldsymbol{U}_k^{\Phi}(\to s') \rangle$ is compatible with the condition $\langle \boldsymbol{T}_k^{\Phi}(u), \boldsymbol{T}_k^{\Phi}(u') \rangle \in D$, as required. □

8. Real names

In this Section, we assume that $\mathbb{P}$ is a **LTF** and $2^{<\omega} \in \mathbb{P}$. It follows by (4.1) that all trees $T[s] = (2^{<\omega})(\to s)$ (see Example 2.2) also belong to $\mathbb{P}$.

Recall that $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ adds a pair of reals $\langle x_{\mathtt{left}}, x_{\mathtt{right}} \rangle \in 2^{\omega} \times 2^{\omega}$.

Arguing in the conditions of Definition 7.1, the goal of the following Theorem 9.3 will be to prove that, for any $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-name c of a real in 2^{ω}, it is forced by the extended forcing $(\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$ that c does not belong to sets of the form $[U]$, where U is a tree in $\mathbb{U}$, **unless** c is a name of one of reals in the E_0-class of one of the generic reals $x_{\mathtt{left}}, x_{\mathtt{right}}$ themselves.

We begin with a suitable notation.

Definition 8.1. A $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-*real name* is a system $\mathbf{c} = \{C_n^i\}_{n<\omega, i<2}$ of sets $C_n^i \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ such that each set $C_n = C_n^0 \cup C_n^1$ is pre-dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ and any conditions $\langle S, S' \rangle \in C_n^0$ and $\langle T, T' \rangle \in C_n^1$ are incompatible in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$.

If a set $G \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic at least over the collection of all sets C_n then we define $\mathbf{c}[G] \in 2^{\omega}$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_n^i \neq \varnothing$. □

Any $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\mathbf{c} = \{C_n^i\}$ induces (can be understood as) a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-name (in the ordinary forcing notation) for a real in 2^{ω}.

Definition 8.2 (actions)**.** Strings in $2^{<\omega}$ can act on names $\mathbf{c} = \{C^i_n\}_{n<\omega, i<2}$ in two ways, related either to conditions or to the output.

If $\sigma, \sigma' \in 2^{<\omega}$ then define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\langle\sigma,\sigma'\rangle \circ \mathbf{c} = \{\langle\sigma,\sigma'\rangle \cdot C^i_n\}$, where $\langle\sigma,\sigma'\rangle \cdot C^i_n = \{\langle\sigma \cdot T, \sigma' \cdot T'\rangle : \langle T, T'\rangle \in C^i_n\}$ for all n, i.

If $\rho \in 2^{<\omega}$ then define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\rho \cdot \mathbf{c} = \{C\rho^i_n\}$, where $C\rho^i_n = C^{1-i}_n$ whenever $n < \mathtt{lh}(\rho)$ and $\rho(n) = 1$, but $C\rho^i_n = C^i_n$ otherwise. $\square$

Both actions are idempotent. The difference between them is as follows. If $G \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic set then $(\langle\sigma,\sigma'\rangle \circ \mathbf{c})[G] = \mathbf{c}[\langle\sigma,\sigma'\rangle \circ G]$, where $\langle\sigma,\sigma'\rangle \circ G = \{\langle\sigma \cdot T, \sigma' \cdot T'\rangle : \langle T, T'\rangle \in G\}$, while $(\rho \cdot \mathbf{c})[G] = \rho \cdot (\mathbf{c}[G])$.

Example 8.3. Define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\dot{\boldsymbol{x}}_{\mathtt{left}} = \{C^i_n\}_{n<\omega, i<2}$ such that each set $C^i_n \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ contains all pairs of the form $\langle T[s], T[t]\rangle$, where $s, t \in 2^{n+1}$ and $s(n) = i$, and a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\dot{\boldsymbol{x}}_{\mathtt{right}} = \{C^i_n\}_{n<\omega, i<2}$ such that accordingly each set $C^i_n \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ contains all pairs $\langle T[s], T[t]\rangle$, where $s, t \in 2^{n+1}$ and now $t(n) = i$. $\square$

Then $\dot{\boldsymbol{x}}_{\mathtt{left}}$, $\dot{\boldsymbol{x}}_{\mathtt{right}}$ are names of the $\mathbb{P}$-generic reals $x_{\mathtt{left}}$, resp., $x_{\mathtt{right}}$, and each name $\sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{left}}$ ($\sigma \in 2^{<\omega}$) induces a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-name of the real $\sigma \cdot (x_{\mathtt{left}}[G])$; the same for $_{\mathtt{right}}$.

9. Direct forcing a real to avoid a tree

Let $\mathbf{c} = \{C^i_n\}$, $\mathbf{d} = \{D^i_n\}$ be $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real names. Say that a condition $\langle T, T'\rangle \in \mathbf{LT} \times_{\mathsf{E}_0} \mathbf{LT}$:

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$, $i = 0, 1$, if $\langle T, T'\rangle \le \langle S, S'\rangle$ for some $\langle S, S'\rangle \in C^i_n$;
- *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \mathtt{lh}(s)$, $\langle T, T'\rangle$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{d} \ne \mathbf{c}$, iff there are strings $s, t \in 2^{<\omega}$, incomparable in $2^{<\omega}$ and such that $\langle T, T'\rangle$ directly forces $s \subset \mathbf{c}$ and $t \subset \mathbf{d}$;
- *directly forces* $\mathbf{c} \notin [U]$, where $U \in \mathbf{PT}$, iff there is a string $s \in 2^{<\omega} \smallsetminus U$ such that $\langle T, T'\rangle$ directly forces $s \subset \mathbf{c}$.

Lemma 9.1. *If $S \in \mathbb{P}$, $\langle R, R'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, and $\mathbf{c}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name, then there exists a tree $S' \in \mathbb{P}$ and a condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \le \langle R, R'\rangle$, such that $S' \subseteq S$ and $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [S']$.*

Proof. Clearly there is a condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \le \langle R, R'\rangle$, which directly forces $u \subset \mathbf{c}$ for some $u \in 2^{<\omega}$ satisfying $\mathtt{lh}(u) > \mathtt{lh}((\mathtt{stem}(S)))$. There is a string $v \in S$, $\mathtt{lh}(v) = \mathtt{lh}(u)$, incomparable with u. The tree $S' = S\restriction_v$ belongs to $\mathbb{P}$, $S' \subseteq S$ by construction, and obviously $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [S']$. $\square$

Lemma 9.2. *If $\mathbf{c}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name, $\sigma \in 2^{<\omega}$, and a condition $\langle R, R'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ directly forces $\sigma \cdot \mathbf{c} \ne \dot{\boldsymbol{x}}_{\mathtt{left}}$, resp., $\sigma \cdot \mathbf{c} \ne \dot{\boldsymbol{x}}_{\mathtt{right}}$, then there is a stronger condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \le \langle R, R'\rangle$, which directly forces resp. $\mathbf{c} \notin [\sigma \cdot T]$, $\mathbf{c} \notin [\sigma \cdot T']$.*

Proof. We just prove the "left" version, as the "right" version can be proved similarly. So let's assume that $\langle R, R'\rangle$ directly forces $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\mathtt{left}}$. There are incomparable strings $u, v \in 2^{<\omega}$ such that $\langle R, R'\rangle$ directly forces $u \subset \sigma \cdot \mathbf{c}$, hence, $\sigma \cdot u \subset \mathbf{c}$ as well, and also directly forces $v \subset \dot{\boldsymbol{x}}_{\mathtt{left}}$. Then by necessity $v \in R$, hence $T = R{\restriction}_v \in \mathbb{P}$, but $u \notin T$. Let $T' = \rho \cdot T$, where $\rho \in 2^{<\omega}$ satisfies $R' = \rho \cdot R$. By definition, the condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ directly forces $\mathbf{c} \notin [\sigma \cdot T]$ (witnessed by $s = \sigma \cdot u$), as required. □

Theorem 9.3. *With the assumptions of Definition 7.1, suppose that* $\mathbf{c} = \{C^i_m\}_{m<\omega, i<2} \in \mathfrak{M}$ *is a* $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$*-real name, and for every* $\sigma \in 2^{<\omega}$ *the set*

$$D_\sigma = \{\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P} : \langle T, T'\rangle \text{ directly forces } \mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{left}} \text{ and } \mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{right}}\}$$

is dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. *Let* $\langle W, W'\rangle \in (\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$ *and* $U \in \mathbb{U}$.

Then there is a stronger condition $\langle V, V'\rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$, $\langle V, V'\rangle \leq \langle W, W'\rangle$, *which directly forces* $\mathbf{c} \notin [U]$.

Proof. By construction, $U = \rho \cdot \boldsymbol{U}^\Phi_K(\to s_0)$, where $K < \omega$ and $\rho, s_0 \in 2^{<\omega}$; we can assume that simply $s_0 = \Lambda$, so that $U = \rho \cdot \boldsymbol{U}^\Phi_K$. Moreover we can assume that $\rho = \Lambda$ as well, so that $U = \boldsymbol{U}^\Phi_K$ (for if not then replace $\mathbf{c}$ with $\rho \cdot \mathbf{c}$).

Further, by Corollary 7.5, we can assume that $W = \sigma \cdot \boldsymbol{U}^\Phi_L(\to t_0) \in \mathbb{U}$ and $W' = \sigma' \cdot \boldsymbol{U}^\Phi_L(\to t'_0) \in \mathbb{U}$, where $L < \omega$, $t_0, t'_0 \in 2^{<\omega}$, $\mathtt{lh}(t_0) = \mathtt{lh}(t'_0)$, and $\sigma, \sigma' \in 2^{<\omega}$. And moreover we can assume that $\sigma = \sigma' = \Lambda$, so that $W = \boldsymbol{U}^\Phi_L(\to t_0)$ and $W' = \boldsymbol{U}^\Phi_L(\to t'_0)$ (for if not then replace $\mathbf{c}$ with $\langle \sigma, \sigma'\rangle \circ \mathbf{c}$).

The indices K, L involved can be either equal or different.

There is an index J such that the multitree $\varphi(J)$ satisfies $K, L \in |\varphi(J)|$ and $h^{\varphi(J)}_L \geq h_0 = \mathtt{lh}(t_0) = \mathtt{lh}(t'_0)$, so that the trees $S_0 = T^{\varphi(J)}_K(0) = \boldsymbol{T}^\Phi_K(0)$,

$$T_0 = T^{\varphi(J)}_L(h_0)(\to t_0) = \boldsymbol{T}^\Phi_L(t_0), \quad T'_0 = T^{\varphi(J)}_L(h_0)(\to t'_0) = \boldsymbol{T}^\Phi_L(t'_0)$$

in $\mathbb{P}$ are defined. Note that $U \subseteq S_0$ and $W \subseteq T_0$, $W' \subseteq T'_0$ under the above assumptions.

Let $\mathscr{D}$ be the set of all multitrees $\varphi \in \mathbf{MT}(\mathbb{P})$ such that $\varphi(J) \preccurlyeq \varphi$ and for every pair $t, t' \in 2^n$, where $n = h^\varphi_L$, such that $t(n-1) \neq t'(n-1)$, the condition $\langle T^\varphi_L(t), T^\varphi_L(t')\rangle$ directly forces $\mathbf{c} \notin [T^\varphi_K(m)]$, where $m = h^\varphi_K$.

Claim 9.4. $\mathscr{D}$ *is dense in* $\mathbf{MT}(\mathbb{P})$ *above* $\varphi(J)$.

Proof. Let a multitree $\psi \in \mathbf{MT}(\mathbb{P})$ satisfy $\varphi(J) \preccurlyeq \psi$; the goal is to define a multitree $\varphi \in \mathscr{D}$, $\psi \preccurlyeq \varphi$. Let $m = h^\psi_K$, $n = h^\psi_L$, $Q = T^\psi_K(m)$, $P = T^\psi_L(n)$.

Case 1: $K \neq L$. Consider any $s \in 2^{m+1}$ and $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$. By Lemma 9.1, there is a tree $S \in \mathbb{P}$ and a condition $\langle R, R'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ such that $S \subseteq Q(\to s)$, $\langle R, R'\rangle \leq \langle P(\to t), P(\to t')\rangle$, and $\langle R, R'\rangle$ directly forces $\mathbf{c} \notin [S]$. By Lemma 4.1(ii),(iv) there are trees $Q_1 \in \mathbf{LC}_{m+1}(\mathbb{P})$ and $P_1 \in \mathbf{LC}_{n+1}(\mathbb{P})$ such that $Q_1 \subseteq_{m+1} Q$, $P_1 \subseteq_{n+1} P$, $Q_1(\to s) = S$ and $\langle P_1(\to t), P_1(\to t')\rangle \leq \langle R, R'\rangle$.

Repeat this procedure so that all strings $s \in 2^{m+1}$ and all pairs of strings $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$ are considered. We obtain trees $Q' \in \mathbf{LC}_{m+1}(\mathbb{P})$

and $P' \in \mathbf{LC}_{n+1}(\mathbb{P})$ such that $Q' \subseteq_{m+1} Q$, $P' \subseteq_{n+1} P$, and if $s \in 2^{m+1}$ and $t, t' \in 2^{n+1}$, $t(n) \neq t'(n)$, the condition $\langle P'(\to t), P'(\to t')\rangle$ directly forces $\mathbf{c} \notin [Q'(\to s)]$ — hence directly forces $\mathbf{c} \notin [Q']$.

Now define a multitree $\varphi \in \mathbf{MT}(\mathbb{P})$ so that $|\varphi| = |\psi|$, $h^\varphi_k = h^\psi_k$ and $\tau^\varphi_k = \tau^\psi_k$ for all $k \notin \{K, L\}$, $h^\varphi_K = m+1$, $h^\varphi_L = n+1$, and $T^\varphi_K(m+1) = P'$, $T^\varphi_L(n+1) = Q'$ as the new elements of the Kth and Lth components. We have $\varphi \in \mathscr{D}$ and $\psi \preccurlyeq \varphi$ by construction. (Use the fact that $P' \subseteq_{n+1} P$ and $Q' \subseteq_{m+1} Q$.)

Case 2: $L = K$, and hence $m = n$ and $P = Q$. Let $h = \mathbf{spl}_n(P)$. Consider any pair $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$. In our assumptions there is a condition $\langle U, U'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle U, U'\rangle \leq \langle T(\to t), T(\to t')\rangle$, which directly forces both $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{left}}$ and $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{right}}$ for any $\sigma \in 2^h$. By Lemma 9.2, there is a stronger condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \leq \langle U, U'\rangle$, which directly forces both $\mathbf{c} \notin [\sigma \cdot T]$ and $\mathbf{c} \notin [\sigma \cdot T']$ still for all $\sigma \in 2^h$. Then as in Case 1, there is a tree $P_1 \in \mathbf{LC}_{n+1}(\mathbb{P})$, $P_1 \subseteq_{n+1} P$, such that $P_1(\to t) \subseteq T$, $P_1(\to t') \subseteq T'$.

We claim that $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [P_1]$, or equivalently, directly forces $\mathbf{c} \notin [P_1(\to s^\frown i)]$ for any $s^\frown i \in 2^{n+1}$ (then $s \in 2^n$). Indeed if $s^\frown i \in 2^{n+1}$ then $P_1(\to s^\frown i) = \sigma \cdot P_1(\to t)$ or $= \sigma \cdot P_1(\to t')$ for some $\sigma \in 2^h$ by the choice of h. Therefore $P_1(\to s^\frown i)$ is a subtree of one of the two trees $\sigma \cdot T$ and $\sigma \cdot T'$. The claim now follows from the choice of $\langle T, T'\rangle$. We conclude that the stronger condition $\langle P_1(\to t), P_1(\to t')\rangle \leq \langle T, T'\rangle$ also directly forces $\mathbf{c} \notin [P_1]$.

Repeat this procedure so that all pairs of strings $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$ are considered. We obtain a tree $P' \in \mathbf{LC}_{n+1}(\mathbb{P})$ such that $P' \subseteq_{n+1} P$, and if $t, t' \in 2^{n+1}$, $t(n) \neq t'(n)$, then $\langle P'(\to t), P'(\to t')\rangle$ directly forces $\mathbf{c} \notin [P']$.

Similar to Case 1, define a multitree $\varphi \in \mathbf{MT}(\mathbb{P})$ so that $|\varphi| = |\psi|$, $h^\varphi_k = h^\psi_k$ and $\tau^\varphi_k = \tau^\psi_k$ for all $k \neq K$, $h^\varphi_K = n+1$, and $T^\varphi_K(n+1) = P'$ as the new element of the $(K = L)$th component. Then $\varphi \in \mathscr{D}$, $\psi \preccurlyeq \varphi$. $\square$ (Claim)

We come back to the proof of Theorem 9.3. The lemma implies that there is an index $j \geq J$ such that the multitree $\varphi(j)$ belongs to $\mathscr{D}$. Let $n = h^{\varphi(j)}_L$, $m = h^{\varphi(j)}_K$. Pick strings $t, t' \in 2^n$ such that $t_0 \subset t$, $t'_0 \subset t'$, $t(n) \neq t'(n)$. Let

$$T = T^{\varphi(j)}_L(t) = \boldsymbol{T}^\Phi_L(t),\ T' = T^{\varphi(j)}_L(t') = \boldsymbol{T}^\Phi_L(t'),\ S = T^{\varphi(j)}_K(m) = \boldsymbol{T}^\Phi_K(m).$$

Then $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \leq \langle T_0, T'_0\rangle$, and $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [S]$.

Consider the condition $\langle V, V'\rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$, where $V = \boldsymbol{U}^\Phi_L(\to t)$ and $V' = \boldsymbol{U}^\Phi_L(\to t')$ belong to $\mathbb{U}$. (Recall that $V = \boldsymbol{U}^\Phi_L(\to t)$ and $V' = \boldsymbol{U}^\Phi_L(\to t')$, and hence $V' = \sigma \cdot V$ for a suitable $\sigma \in 2^{<\omega}$.) By construction we have both $\langle V, V'\rangle \leq \langle W, W'\rangle$ (as $t_0 \subseteq t, t'$) and $\langle V, V'\rangle \leq \langle T, T'\rangle \leq \langle T_0, T'_0\rangle$. Therefore $\langle V, V'\rangle$ directly forces $\mathbf{c} \notin [S]$. And finally, we have $U \subseteq T^{\varphi(j)}_K(m) = S$, so that $\langle V, V'\rangle$ directly forces $\mathbf{c} \notin [U]$, as required. $\square$ (Theorem 9.3)

10. Jensen's forcing

In this section, **we argue in $\mathbf{L}$, the constructible universe.** Let $\le_{\mathbf{L}}$ be the canonical wellordering of $\mathbf{L}$.

Definition 10.1 (in $\mathbf{L}$)**.** Following the construction in [7, Section 3] *mutatis mutandis*, define, by induction on $\xi < \omega_1$, a countable **LTF** $\mathbb{U}_\xi \subseteq \mathbf{LT}$ as follows.

Let $\mathbb{U}_0$ consist of all trees of the form $T[s]$, see Example 2.2.

Suppose that $0 < \lambda < \omega_1$, and countable **LTF**s $\mathbb{U}_\xi \subseteq \mathbf{LT}$ are defined for $\xi < \lambda$. Let $\mathfrak{M}_\lambda$ be the least model $\mathfrak{M}$ of $\mathbf{ZFC}'$ of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, containing $\{\mathbb{U}_\xi\}_{\xi<\lambda}$ and such that $\lambda < \omega_1^{\mathfrak{M}}$ and all sets $\mathbb{U}_\xi$, $\xi < \lambda$, are countable in $\mathfrak{M}$. Then $\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{U}_\xi$ is countable in $\mathfrak{M}$, too. Let $\{\varphi(j)\}_{j<\omega}$ be the $\le_{\mathbf{L}}$-least sequence of multitrees $\varphi(j) \in \mathbf{MT}(\mathbb{P}_\lambda)$, $\preccurlyeq$-increasing and generic over $\mathfrak{M}_\lambda$. Define $\mathbb{U}_\lambda = \mathbb{U}$ as in Definition 7.1. This completes the inductive step.

Let $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{U}_\xi$. $\square$

Proposition 10.2 (in $\mathbf{L}$)**.** *The sequence $\{\mathbb{U}_\xi\}_{\xi<\omega_1}$ belongs to $\varDelta^{\mathrm{HC}}_1$.* $\square$

Lemma 10.3 (in $\mathbf{L}$)**.** *If a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathbb{P}_\xi$ is pre-dense in $\mathbb{P}_\xi$ then it remains pre-dense in $\mathbb{P}$. Therefore if $\xi < \omega_1$ then $\mathbb{U}_\xi$ is pre-dense in $\mathbb{P}$.*

If a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathbb{P}_\xi \times_{\mathsf{E}_0} \mathbb{P}_\xi$ is pre-dense in $\mathbb{P}_\xi \times_{\mathsf{E}_0} \mathbb{P}_\xi$ then it is pre-dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$.

Proof. By induction on $\lambda \ge \xi$, if D is pre-dense in $\mathbb{P}_\lambda$ then it remains pre-dense in $\mathbb{P}_{\lambda+1} = \mathbb{P}_\lambda \cup \mathbb{U}_\lambda$ by Lemma 7.6. Limit steps are obvious. To prove the second claim note that $\mathbb{U}_\xi$ is dense in $\mathbb{P}_{\xi+1}$ by Lemma 7.3, and $\mathbb{U}_\xi \in \mathfrak{M}_{\xi+1}$.

To prove the last claim use Lemma 7.7. $\square$

Lemma 10.4 (in $\mathbf{L}$)**.** *If $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$ then the set W_X of all ordinals $\xi < \omega_1$ such that $\langle \mathbf{L}_\xi\,; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X \rangle$ and $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ is unbounded in ω_1. More generally, if $X_n \subseteq \mathrm{HC}$ for all n then the set W of all ordinals $\xi < \omega_1$, such that $\langle \mathbf{L}_\xi\,; \{X_n \cap \mathbf{L}_\xi\}_{n<\omega} \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; \{X_n\}_{n<\omega} \rangle$ and $\{X_n \cap \mathbf{L}_\xi\}_{n<\omega} \in \mathfrak{M}_\xi$, is unbounded in ω_1.*

Proof. Let $\xi_0 < \omega_1$. Let M be a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing ξ_0, ω_1, X, and such that $M \cap \mathrm{HC}$ is transitive. Let $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and let $\xi = \phi(\omega_1)$. Then $\xi_0 < \xi < \lambda < \omega_1$ and $\phi(X) = X \cap \mathbf{L}_\xi$ by the choice of M. It follows that $\langle \mathbf{L}_\xi\,; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X \rangle$. Moreover, ξ is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\xi$. We conclude that $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ since $X \cap \mathbf{L}_\xi \in \mathbf{L}_\lambda$ by construction.

The second claim does not differ much: we start with a model M containing both the whole sequence $\{X_n\}_{n<\omega}$ and each particular X_n, and so on. $\square$

Corollary 10.5 (compare to [7], Lemma 6). *The forcing notions* $\mathbb{P}$ *and* $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ *satisfy CCC in* $\mathbf{L}$.

Proof. Suppose that $A \subseteq \mathbb{P}$ is a maximal antichain. By Lemma 10.4, there is an ordinal ξ such that $A' = A \cap \mathbb{P}_\xi$ is a maximal antichain in $\mathbb{P}_\xi$ and $A' \in \mathfrak{M}_\xi$. But then A' remains pre-dense, therefore, still a maximal antichain, in the whole set $\mathbb{P}$ by Lemma 10.3. It follows that $A = A'$ is countable. □

11. The model

We view the sets $\mathbb{P}$ and $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ (Definition 10.1) as forcing notions over $\mathbf{L}$.

Lemma 11.1 (compare to Lemma 7 in [7]). *A real* $x \in 2^\omega$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$ *iff* $x \in Z = \bigcap_{\xi<\omega_1^{\mathbf{L}}} \bigcup_{U \in \mathbb{U}_\xi} [U]$.

Proof. If $\xi < \omega_1^{\mathbf{L}}$ then $\mathbb{U}_\xi$ is pre-dense in $\mathbb{P}$ by Lemma 10.3, therefore any real $x \in 2^\omega$ $\mathbb{P}$-generic over $\mathbf{L}$ belongs to $\bigcup_{U \in \mathbb{U}_\xi} [U]$.

To prove the converse, suppose that $x \in Z$ and prove that x is $\mathbb{P}$-generic over $\mathbf{L}$. Consider a maximal antichain $A \subseteq \mathbb{P}$ in $\mathbf{L}$; we have to prove that $x \in \bigcup_{T \in A} [T]$. Note that $A \subseteq \mathbb{P}_\xi$ for some $\xi < \omega_1^{\mathbf{L}}$ by Corollary 10.5. But then every tree $U \in \mathbb{U}_\xi$ satisfies $U \subseteq^{\mathtt{fin}} \bigcup A$ by Lemma 7.6, so that $\bigcup_{U \in \mathbb{U}_\xi} [U] \subseteq \bigcup_{T \in A} [T]$, and hence $x \in \bigcup_{T \in A} [T]$, as required. □

Corollary 11.2 (compare to Corollary 9 in [7]). *In any generic extension of* $\mathbf{L}$, *the set of all reals in* 2^ω $\mathbb{P}$*-generic over* $\mathbf{L}$ *is* Π_1^{HC} *and* Π_2^1.

Proof. Use Lemma 11.1 and Proposition 10.2. □

Definition 11.3. From now on, we assume that $G \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is a set $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic over $\mathbf{L}$, so that the intersection $X = \bigcap_{\langle T,T' \rangle \in G} [T] \times [T']$ is a singleton $X_G = \{\langle x_{\mathtt{left}}[G], x_{\mathtt{right}}[G] \rangle\}$. □

Compare the next lemma to Lemma 10 in [7]. While Jensen's forcing notion in [7] guarantees that there is a single generic real in the extension, the forcing notion $\mathbb{P}$ we use adds a whole E_0-class (a countable set) of generic reals!

Lemma 11.4 (under the assumptions of Definition 11.3). *If* $y \in \mathbf{L}[G] \cap 2^\omega$ *then* y *is a* $\mathbb{P}$*-generic real over* $\mathbf{L}$ *iff* $y \in [x_{\mathtt{left}}[G]]_{\mathsf{E}_0} \cup [x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$.

Recall that $[x]_{\mathsf{E}_0} = \{\sigma \cdot x : \sigma \in 2^{<\omega}\}$.

Proof. The reals $x_{\mathtt{left}}[G]$, $x_{\mathtt{right}}[G]$ are separately $\mathbb{P}$-generic (see Remark 5.1). It follows that any real $y = \sigma \cdot x_{\mathtt{left}}[G] \in [x_{\mathtt{left}}[G]]_{\mathsf{E}_0}$ or $y = \sigma \cdot x_{\mathtt{right}}[G] \in [x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$ is $\mathbb{P}$-generic as well since the forcing $\mathbb{P}$ is by definition invariant under the action of any $\sigma \in 2^{<\omega}$.

To prove the converse, suppose towards the contrary that there is a condition $\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ and a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\mathbf{c} = \{C_n^i\}_{n<\omega,\, i=0,1} \in \mathbf{L}$

such that $\langle T,T'\rangle$ $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-forces that $\mathbf{c}$ is $\mathbb{P}$-generic while $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ forces both formulas $\mathbf{c}\neq\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{left}}$ and $\mathbf{c}\neq\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{left}}$ for all $\sigma\in 2^{<\omega}$.

Let $C_n=C_n^0\cup C_n^1$, this is a pre-dense set in $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$. It follows from Lemma 10.4 that there exists an ordinal $\lambda<\omega_1$ such that each set $C'_n=C_n\cap(\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda)$ is pre-dense in $\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda$, and the sequence $\{C'_{ni}\}_{n<\omega,\,i=0,1}$ belongs to $\mathfrak{M}_\lambda$, where $C'_{ni}=C'_n\cap C^i_n$ — then C'_n is pre-dense in $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ too, by Lemma 10.3. Therefore we can assume that in fact $C_n=C'_n$, that is, $\mathbf{c}\in\mathfrak{M}_\lambda$ and $\mathbf{c}$ is a $(\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda)$-real name.

Further, as $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ forces that $\mathbf{c}\neq\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{left}}$ and $\mathbf{c}\neq\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{right}}$, the set $D(\sigma)$ of all conditions $\langle S,S'\rangle\in\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ which directly force $\mathbf{c}\neq\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{left}}$ and $\mathbf{c}\neq\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{right}}$, is dense in $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ — for every $\sigma\in 2^{<\omega}$. Therefore, still by Lemma 10.4, we may assume that the same ordinal λ as above satisfies the following: each set $D'(\sigma)=D(\sigma)\cap(\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda)$ is dense in $\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda$.

Applying Theorem 9.3 with $\mathbb{P}=\mathbb{P}_\lambda$, $\mathbb{U}=\mathbb{U}_\lambda$, and $\mathbb{P}\cup\mathbb{U}=\mathbb{P}_{\lambda+1}$, we conclude that for each tree $U\in\mathbb{U}_\lambda$ the set Q_U of all conditions $\langle V,V'\rangle\in\mathbb{P}_{\lambda+1}\times_{\mathsf{E}_0}\mathbb{P}_{\lambda+1}$ which directly force $\mathbf{c}\notin[U]$, is dense in $\mathbb{P}_{\lambda+1}\times_{\mathsf{E}_0}\mathbb{P}_{\lambda+1}$. As obviously $Q_U\in\mathfrak{M}_{\lambda+1}$, we further conclude that Q_U is pre-dense in the whole forcing $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ by Lemma 10.3. This implies that $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ forces $\mathbf{c}\notin\bigcup_{U\in\mathbb{U}_\lambda}[U]$, hence, forces that $\mathbf{c}$ is not $\mathbb{P}$-generic, by Lemma 11.1. But this contradicts to the choice of $\langle T,T'\rangle$. $\square$

Corollary 11.5. *The set $[x_{\mathtt{left}}[G]]_{\mathsf{E}_0}\cup[x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$ is Π^1_2 set in $\mathbf{L}[G]$. Therefore the 2-element set $\{[x_{\mathtt{left}}[G]]_{\mathsf{E}_0},[x_{\mathtt{right}}[G]]_{\mathsf{E}_0}\}$ is OD in $\mathbf{L}[G]$.* $\square$

Corollary 11.6. *The E_0-classes $[x_{\mathtt{left}}[G]]_{\mathsf{E}_0}$, $[x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$ are disjoint.*

Proof. Corollary 5.4 implies $x_{\mathtt{left}}[G]\;\not\mathsf{E}_0\;x_{\mathtt{right}}[G]$. $\square$

Lemma 11.7 (still under the assumptions of Definition 11.3)**.** *Neither of the two E_0-classes $[x_{\mathtt{left}}[G]]_{\mathsf{E}_0}$, $[x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$ is OD in $\mathbf{L}[G]$.*

Proof. Suppose towards the contrary that there is a condition $\langle T,T'\rangle\in G$ and a formula $\vartheta(x)$ with ordinal parameters such that $\langle T,T'\rangle$ $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-forces that $\vartheta([\dot{\boldsymbol{x}}_{\mathtt{left}}]_{\mathsf{E}_0})$ but $\neg\vartheta([\dot{\boldsymbol{x}}_{\mathtt{right}}]_{\mathsf{E}_0})$. However both the formula and the forcing are invariant under actions of strings in $2^{<\omega}$. In particular if $\sigma\in 2^{<\omega}$ then $\langle\sigma\cdot T,\sigma\cdot T'\rangle$ still $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-forces $\vartheta([\dot{\boldsymbol{x}}_{\mathtt{left}}]_{\mathsf{E}_0})$ and $\neg\,\vartheta([\dot{\boldsymbol{x}}_{\mathtt{right}}]_{\mathsf{E}_0})$. We can take σ which satisfies $T'=\sigma\cdot T$; thus $\langle T',T\rangle$ still $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-forces $\vartheta([\dot{\boldsymbol{x}}_{\mathtt{left}}]_{\mathsf{E}_0})$ and $\neg\vartheta([\dot{\boldsymbol{x}}_{\mathtt{right}}]_{\mathsf{E}_0})$.[5] However $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ is symmetric with respect to the left-right exchange, which implies that conversely $\langle T',T\rangle$ has to force $\vartheta([\dot{\boldsymbol{x}}_{\mathtt{right}}]_{\mathsf{E}_0})$ and $\neg\,\vartheta([\dot{\boldsymbol{x}}_{\mathtt{left}}]_{\mathsf{E}_0})$. The contradiction proves the lemma. $\square$

$\square$ (Theorem 1.1)

12. Conclusive remarks

(I) One may ask whether other Borel equivalence relations E admit results similar to Theorem 1.1. Fortunately this question can be easily solved on the base of the Glimm – Effros dichotomy theorem [4].

[5] This is the argument which does not go through for the full product $\mathbb{P}\times\mathbb{P}$.

Corollary 12.1. *The following is true in the model of Theorem 1.1. Let* E *be a Borel equivalence relation on* ω^ω *coded in* $\mathbf{L}$*. Then there exists an OD pair of* E*-equivalence classes* $\{[x]_\mathsf{E}, [y]_\mathsf{E}\}$ *such that neither of the classes* $[x]_\mathsf{E}$, $[y]_\mathsf{E}$ *is separately OD, iff* E *is not smooth.*

Proof. Suppose first that E is smooth. By the Shoenfield absoluteness theorem, the smoothness can be witnessed by a Borel map $\vartheta : \omega^\omega \to \omega^\omega$ coded in $\mathbf{L}$, hence, ϑ is OD itself. If $p = \{[x]_\mathsf{E}, [y]_\mathsf{E}\}$ is OD in the extension then so is the 2-element set $R = \{\vartheta(z) : z \in [x]_\mathsf{E} \cup [y]_\mathsf{E}\} \subseteq \omega^\omega$, whose both elements (reals), say p_x and p_y, are OD by obvious reasons. Then finally $[x]_\mathsf{E} = \vartheta^{(-1)}(p_x)$ and $[y]_\mathsf{E} = \vartheta^{(-1)}(p_y)$ are OD as required.

Now let E be non-smooth. Then by Shoenfield and the Glimm – Effros dichotomy theorem in [4], there is a continuous, coded by some $r \in \omega^\omega \cap \mathbf{L}$, hence, OD, reduction $\vartheta : 2^\omega \to \omega^\omega$ of E_0 to E, so that we have $a \mathrel{\mathsf{E}_0} b$ iff $\vartheta(a) \mathrel{\mathsf{E}} \vartheta(b)$ for all $a, b \in 2^\omega$. Let, by Theorem 1.1, $\{[a]_{\mathsf{E}_0}, [b]_{\mathsf{E}_0}\}$ be a Π^1_2 pair of non-OD E_0-equivalence classes. By the choice of ϑ, one easily proves that $\{[\vartheta(a)]_\mathsf{E}, [\vartheta(b)]_\mathsf{E}\}$ is a $\Pi^1_2(r)$ pair of non-OD E-equivalence classes. $\square$

(II) One may ask what happens with the Groszek – Laver pairs of sets of reals in better known models. For some of them the answer tends to be in the negative. Consider *e.g.* the Solovay model of **ZFC** in which all projective sets of reals are Lebesgue measurable [14]. Arguing in the Solovay model, let $\{X, Y\}$ be an OD set, where $X, Y \subseteq 2^\omega$. Then the set of *four* sets $X \smallsetminus Y$, $Y \smallsetminus X$, $X \cap Y$, $2^\omega \smallsetminus (X \cup Y)$ is still OD, and hence we have an OD equivalence relation E on 2^ω with four (or fewer if say $X \subseteq Y$) equivalence classes. By a theorem of [8] [6], either E admits an OD reduction $\vartheta : 2^\omega \to 2^{<\omega_1}$ to equality on $2^{<\omega_1}$ or E_0 admits a continuous reduction to E. The "or" option fails since E has finitely many classes.

The "either" option leads to a finite (not more than 4 elements) OD set $R = \mathtt{ran}\,\vartheta \subseteq 2^{<\omega_1}$. An easy argument shows that then every $r \in R$ is OD, and hence so is the corresponding E-class $\vartheta^{-1}(r)$. It follows that X, Y themselves are OD.

Question 12.2. Is it true in the Solovay model that every *countable* OD set $W \subseteq \mathscr{P}(\omega^\omega)$ *of sets of reals* contains an OD element $X \in W$ (a set of reals)? $\square$

An uncountable counterexample readily exists, for take the set of all non-OD sets of reals. As for sets $W \subseteq \omega^\omega$, any countable OD set of reals in the Solovay model consists of OD elements, *e.g.* by the result mentioned in Footnote 6.

(III) One may ask whether a forcing similar to $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ with respect to the results in Section 11, exists in ground models other than $\mathbf{L}$ or $\mathbf{L}[x]$,

[6] To replace the following brief argument, one can also refer to a result by Stern implicit in [15]: in the Solovay model, if an OD equivalence relation E has at least one non-OD equivalence class then there is a pairwise E-inequivalent perfect set.

$x \in 2^\omega$. Some coding forcing constructions with perfect trees do exist in such a general frameworks, see [1, 10].

Acknowledgements. The authors thank Ali Enayat for the interest in the problem and helpful remarks. The authors thank the anonymous referee for many important suggestions that helped to improve the text.

References

1. J. Bagaria and V. Kanovei, *On coding uncountable sets by reals*, Mathematical Logic Quarterly, 56(4): 409–424, 2010.
2. Ali Enayat, *On the Leibniz-Mycielski axiom in set theory*, Fundam. Math., 181(3): 215–231, 2004.
3. M. Groszek and R. Laver, *Finite groups of OD-conjugates*, Period. Math. Hung., 18: 87–97, 1987.
4. L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc., 3(4): 903–928, 1990.
5. L. A. Harrington, D. Marker, and S. Shelah, *Borel orderings*, Trans. Amer. Math. Soc., 310(1): 293–302, 1988.
6. Thomas Jech, *Set theory*, Berlin: Springer, the third millennium revised and expanded edition, 2003.
7. Ronald Jensen, *Definable sets of minimal degree*, Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968, pp. 122-128, 1970.
8. Vladimir Kanovei, *An Ulm-type classification theorem for equivalence relations in Solovay model*, J. Symbolic Logic, 62(4): 1333–1351, 1997.
9. Vladimir Kanovei, *Borel equivalence relations. Structure and classification*, Providence, RI: American Mathematical Society (AMS), 2008.
10. V. Kanovei and V. Lyubetsky, *An effective minimal encoding of uncountable sets*, Siberian Mathematical Journal, 52(5): 854–863, 2011.
11. V. Kanovei and V. Lyubetsky, *A countable definable set of reals containing no definable elements*, ArXiv e-prints, 1408.3901, August 2014.
12. V. Kanovei and V. Lyubetsky, *A definable E_0-class containing no definable elements*, Archive for Mathematical Logic, 2015, 54, 5, pp. 711–723.
13. Vladimir Kanovei, Martin Sabok, and Jindřich Zapletal, *Canonical Ramsey theory on Polish space*, Cambridge: Cambridge University Press, 2013.
14. R.M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. Math. (2), 92: 1–56, 1970.
15. J. Stern, *On Lusin's restricted continuum problem*, Ann. Math. (2), 120: 7–37, 1984.

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran-Iran
E-mail address: golshani.m@gmail.com

IITP RAS and MIIT, Moscow, Russia
E-mail address: kanovei@googlemail.com

IITP RAS, Moscow, Russia
E-mail address: lyubetsk@iitp.ru